Convergency and Divergency of series

An expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ in which each term is formed according to some definite law is called a series.

Finite and Infinite series

A series is said to be finite when it consists of finite (or definite) number of terms.

A series is said to be infinite when it consists of infinite (or indefinite) number of terms.

We shall generally denote an infinite series by $u_1 + u_2 + u_3 + \cdots + u_n + \cdots \infty$ or $\sum_{n \geq 1}^{\infty} u_n$ or $\sum u_n$

Convergent series:- A series is said to be convergent if the sum of first n terms tends to a finite limit as $n \to \infty$

i.e. S_n tends to a finite limit S, the series is said to be convergent.

Example:- Consider the series

$$1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n} + \cdots \text{ to } \infty$$

$$\text{Here } S_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}}$$

Here first term $= 1$ and c.r. $= \frac{1}{3}$

$$\therefore S_n = \frac{1\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}} = \frac{1 - \frac{1}{3^n}}{\frac{3-1}{3}} = \frac{3}{2}\left(1 - \frac{1}{3^n}\right)$$

Now taking limit $n \to \infty$, we have

(v) $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{3}{2}\left(1-\frac{1}{3^n}\right) = \frac{3}{2}(1-0) = \frac{3}{2}$ = a finite quantity

i.e. when $n\to\infty$, $S_n \to \frac{3}{2}$, so, the series is convergent.

Divergent series:- A series is said to be divergent if the sum of first n terms tends to $+\infty$ or $-\infty$ as $n\to\infty$

i.e. if $\lim_{n\to\infty} S_n = +\infty$ or $\lim_{n\to\infty} S_n = -\infty$, then the series is divergent.

Example. consider the series

$$2+4+6+8+ \cdots$$

Here first term $=2$, common difference $=2$

$$\therefore S_n = \frac{n}{2}\{2\cdot 2+(n-1)\cdot 2\} = \frac{n}{2}(2+2n) = n(n+1)$$

Now taking limit $n\to\infty$, we get

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} n(n+1) = \infty$$

$\therefore$ when $n\to\infty$, $S_n\to\infty$. So the series is dgt.

~~Oscillating series:- A series is said to be oscillating if the sum of first n terms neither tends~~ ~~to~~

Oscillating series:- If the sum of ~~does not~~ n terms neither tends to a finite quantity nor to $+\infty$ or $-\infty$ as $n\to\infty$, then the series is said to be oscillating. ~~series~~

Ex:- Let us consider a series

$1-1+1-1+1-1+1-1+ \cdots$

$\therefore S_n = 1-1+1-1+1-1+\cdots$ to n terms

$= (1-1)+(1-1)+(1-1)+\cdots$ to n terms $=0$, when n is even.

Again, $S_n = 1-(1-1)-(1-1)-(1-1)-\cdots$ to n terms $=1$, when n is odd.

$\therefore$ S_n does not tend to any limit finite or infinite. Its sum oscillates between ~~...~~ 0 and 1 according as n is even or odd.

Article **P-series test or Auxiliary series or Standard ~~...~~ series**

Thm: ~~The~~ The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots+\frac{1}{n^p}+\cdots$ is convergent if $p>1$ and divergent if $p\le 1$

proof:- case (1) If $p>1$

The given series can be written as

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\cdots\infty$$

$$\frac{1}{1^p}=\frac{1}{1}$$

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}=\frac{2^2}{2^{2p}}$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\cdots+\frac{1}{15^p}<\frac{8}{8^p}=\frac{2^3}{2^{3p}}=\left(\frac{2}{2^p}\right)^3$$

$\cdots\cdots$

$\cdots\cdots$

④ Adding all

$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots\infty < 1+\frac{2}{2^p}+\left(\frac{2}{2^p}\right)^2+\left(\frac{2}{2^p}\right)^3+\cdots$

The right hand side is a G.P series whose c.r $=\frac{2}{2^p}$ which is <1 ($\because p>1$)

so it is convergent.

Hence $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots\infty$ is also conv.

<u>Case ii</u> If $p=1$

The series becomes.

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\infty$

$1+\frac{1}{2}=1+\frac{1}{2}$

$\frac{1}{3}+\frac{1}{4}>\frac{1}{4}+\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$

$\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}>\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{4}{8}=\frac{1}{2}$

$\frac{1}{9}+\frac{1}{10}+\cdots+\frac{1}{16}>\frac{8}{16}=\frac{1}{2}$

$\cdots\cdots$

Adding all, we get

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

or $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$

[$\because 1=\frac{1}{2}+\frac{1}{2}$]

The right side is div. because each term is $\frac{1}{2}$

$\therefore 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$ is also divergence

Case (iii) If $p < 1$

$\frac{1}{1^p} = \frac{1}{1}$

$\frac{1}{2^p} > \frac{1}{2}$

$\frac{1}{3^p} > \frac{1}{3}$

$\frac{1}{4^p} > \frac{1}{4}$

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Adding all

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots > 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

The right side is divergent on account of case (ii). $\therefore \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$ is also divergent.

Example:- (i) The series $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$ converges, [$\because$ here $p = 2 > 1$]

(ii) The series $\sum \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots$ diverges,

$\because$ here $p = \frac{1}{2} < 1$

(iii) The series

$$\frac{1}{1^\pi} + \frac{1}{2^\pi} + \frac{1}{3^\pi} + \dots \text{ converges}$$

(here $p = \pi = \frac{22}{7} > 1$)

(iv) The series $\frac{1}{1^{3/2}} + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots$ (converges)

(here $p = \frac{3}{2} > 1$)